

Deformations from a given Kähler metric to a twisted cscK metric

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1 Introduction

In 1950's, E. Calabi(cf. [2],[3]) has raised the famous Calabi's conjecture stating that there exists a unique Kähler metric in any given Kähler class whose Ricci form is any given 2-form representing the first chern class. This conjecture was later proved in 1970's by the celebrated works of S. T. Yau([25]), E. Calabi([4]) and T. Aubin([1]) using continuity method to solve the complex Monge-Ampère equation. In particular, when the first chern class is zero or negative, their works imply the existence of Kähler-Einstein metrics. When the first chern class is positive, Tian has made contributions towards understanding precisely when a solution exists([22]).

In 1980's, E. Calabi(cf. [5][6]) proposed a broader program aiming to find the extremal metrics as the generalization of Kähler-Einstein metrics in an arbitrary Kähler class. As a special case of extremal metrics, the existence problem of the constant scalar curvature Kähler(cscK) metrics fits into a general picture of symplectic geometry as described by S. K. Donaldson([13]). It was well known by now that the existence of Kähler-Einstein metrics or cscK metrics was equivalent to some notion of "stability" in algebraic geometry (c.f. Yau-Tian-Donaldson conjecture([24], [22] and [12])). Recently, this conjecture was settled in the Fano case by the crucial contributions of Chen-Donaldson-Sun (cf. [8], [9] and [10]).

In a series of remarkable work ([14],[15], [16] and [17]), S. K. Donaldson proved the existence of cscK metric on a K-stable toric surface. Very little was known for a general Kähler class in higher dimensions. Recently, X. Chen initiates a new program attacking the existence problem of cscK metrics via a new continuity path in [7], which connects the usual cscK metric equation with a second order elliptic equation. As in [7], for a positive closed (1,1)-form χ , we define the Kähler metric ω_φ satisfying

$$t(R_\varphi - \underline{R}) - (1-t)(\text{tr}_\varphi \chi - \underline{\chi}) = 0 \quad (1)$$

the twisted cscK metric, where R_φ denote the scalar curvature of ω_φ , $\underline{R} = \frac{[c_1(M)][\omega]^{[n-1]}}{[\omega]^{[n]}}$ and $\underline{\chi} = \frac{[\chi][\omega]^{[n-1]}}{[\omega]^{[n]}}$. In his same paper, X. Chen also showed the openness of this path when $0 < t < 1$. And in a subsequent paper [11], X.Chen, M. Păun and Y. Zeng used the openness result at $t = 1$ to give a new proof of the uniqueness theorem of extremal metrics. Similar

notions of twisted cscK metrics could also be found in earlier papers of J. Fine [18], J. Stoppa [21] and Lejmi-Székelyhidi [20].

In this paper, we'll prove the openness of the new continuity path introduced in [7] at $t = 0$. And it adds further evidence that the path is the right one to work on. Following from a simple observation, we could choose $\chi = \omega$ in (1) such that (1) always has trivial solution at $t = 0$. Our purpose of this paper is to prove the following main theorem:

Theorem 1.1. *Suppose (M, ω) is a closed Kähler manifold. Then, for any $t > 0$ sufficiently small, there exist a unique smooth Kähler metric ω_{φ_t} such that*

$$t(R_{\varphi_t} - \underline{R}) - (1 - t)(\text{tr}_{\varphi_t} \omega - n) = 0. \quad (2)$$

Notice here, when $t > 0$, (2) is a 4th order nonlinear elliptic equation while at $t = 0$, we get a second order equation. Thus, it's not clear which function space we should choose if we want to apply the inverse function theorem. Fortunately, if we denote $\varphi_0 = 0$, then

$$t(R_{\varphi_0} - \underline{R}) - (1 - t)(\text{tr}_{\varphi_0} \omega - n) = t(R_{\varphi_0} - \underline{R}) \rightarrow 0, \text{ as } t \rightarrow 0$$

in any $C^k(M)$ norm. It suggests that φ_0 is very close to a twisted cscK metric when $t > 0$ sufficiently small. Thus, if we take φ_0 as base point and apply the inverse function theorem at $\varphi = \varphi_0$, there's a slight chance that it contains "0" in its neighborhood of image of φ_0 where every element has a pre-image. However, later we find out that the radius of neighbourhood of $t(R_{\varphi_0} - \underline{R})$ which has pre-images decreases faster than t^2 while "0" lies in only the radius t neighborhood.

To overcome this difficulty, we'll first introduce basic notions in Section 2 and reduce (2) from a 4th order equation to a second order equation

$$r\theta_\varphi + \varphi = 0. \quad (3)$$

Then in Section 3 we could expand the above equation in power series of r and collect the same order terms of r to see possible ways of cancelations. Then we could choose φ_1 closer to the critical point than φ_0 as shown in Lemma 3.1. Namely, we choose φ_1 such that "0" is in the r^4 neighborhood of $r\theta_{\varphi_1} + \varphi_1$ in C^α space. And in Section 4, by intense calculations, we show that the radius of neighborhood of $r\theta_{\varphi_1} + \varphi_1$ which has pre-images is greater than $r^{3+\epsilon}$ for some $\epsilon > 0$. Eventually "0" will fall into the $r^{3+\epsilon}$ neighborhood of image of φ_1 . Thus, it has a pre-image.

Without further notice, the "C" in each estimate means a constant depending on the complex dimension n , the background metric ω , the topological constant \underline{R} and $0 < \alpha < 1$ unless specified.

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2 Preliminary

Suppose (M, ω) is a closed Kähler manifold. Denote the space of normalized smooth Kähler potentials as

$$\mathcal{H}_\omega = \{\varphi \in C^\infty(M) | \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \int_M \varphi \omega^n = 0\}. \quad (4)$$

For $\varphi \in \mathcal{H}_\omega$, we denote R_φ the scalar curvature of ω_φ and $\underline{R} = \frac{[c_1(M)][\omega]^{[n-1]}}{[\omega]^{[n]}}$.

In [7], Chen has introduced a continuity path in \mathcal{H}_ω for a closed positive $(1,1)$ -form χ as

$$t(R_\varphi - \underline{R}) - (1-t)(\text{tr}_\varphi \chi - \underline{\chi}) = 0, \quad (5)$$

where $\underline{\chi} = \frac{[\chi][\omega]^{[n-1]}}{[\omega]^{[n]}}$.

In particular, as described in the introduction, we could simply choose the closed positive $(1,1)$ -form to be ω . Thus, (5) always has a trivial solution at $t = 0$. As in defining the Futaki invariant in [23], we could solve the Laplacian equation for any $\varphi \in \mathcal{H}_\omega$

$$\Delta_\varphi f = R_\varphi - \underline{R}. \quad (6)$$

We denote the solution of (6) as θ_φ with the normalization $\int_M \theta_\varphi \omega^n = 0$. Since

$$R_\varphi - \underline{R} = \text{tr}_\varphi(\text{Ric}_\varphi - \text{Ric}(\omega) + \text{Ric}(\omega) - \underline{R}\omega - \underline{R}\sqrt{-1}\partial\bar{\partial}\varphi), \quad (7)$$

$$= \Delta_\varphi(-\log \frac{\omega_\varphi^n}{\omega^n} - \underline{R}\varphi) + \text{tr}_\varphi(\text{Ric}(\omega) - \underline{R}\omega), \quad (8)$$

we have

$$\theta_\varphi = -\log \frac{\omega_\varphi^n}{\omega^n} - \underline{R}\varphi + \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega^n + P_\varphi, \quad (9)$$

where P_φ is determined by

$$\Delta_\varphi P_\varphi = \text{tr}_\varphi(\text{Ric}(\omega) - \underline{R}\omega), \int_M P_\varphi \omega^n = 0. \quad (10)$$

Given the notion of θ_φ above, we could reduce the continuity path equation (5) with $\chi = \omega$ from a 4th order PDE to a Monge-Ampère type of equation as

$$t\theta_\varphi + (1-t)\varphi = 0. \quad (11)$$

Following the discussion above, Theorem 1.1 will be an easy corollary of the following theorem:

Theorem 2.1. *Suppose (M, ω) is a closed Kähler manifold. Then, for any $r > 0$ sufficiently small, there exists a unique $\varphi_r \in \mathcal{H}_\omega$ such that*

$$r\theta_{\varphi_r} + \varphi_r = 0. \quad (12)$$

In our paper, we'll repeatedly use schauder estimate of Laplacian equation. Thus, let's introduce it here as the following Lemma:

Lemma 2.2. *If $\varphi \in C^{2,\alpha}(M)$ with $\|\varphi\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$ and $u \in C^{2,\alpha}(M)$ statisfies*

$$\Delta_\varphi u = f, \int_M u \omega^n = 0 \quad (13)$$

for some $f \in C^\alpha(M)$. Then

$$\|u\|_{C^{2,\alpha}(M)} \leq C\|f\|_{C^\alpha(M)}. \quad (14)$$

Proof. Proof of Lemma 2.2. By Schauder estimate [19], we can get

$$\|u\|_{C^{2,\alpha}(M)} \leq C(\|u\|_{L^\infty(M)} + \|f\|_{C^\alpha(M)}). \quad (15)$$

To bound $\|u\|_{L^\infty(M)}$, we first multiply u on both hand sides of (13), integrate against ω_φ^n , and we get that

$$\int_M |\nabla u|_\varphi^2 \omega_\varphi^n = \int_M -f u \omega_\varphi^n \leq C\|f\|_{L^\infty(M)} \|u\|_{L^2(M,\omega)}. \quad (16)$$

On the other hand,

$$\int_M |\nabla u|_\varphi^2 \omega_\varphi^n \geq \frac{1}{C} \int_M |\nabla u|^2 \omega^n \geq \frac{1}{C} \|u\|_{L^2(M,\omega)}^2. \quad (17)$$

Thus, combining the above two inequalities, we can get

$$\|u\|_{L^2(M,\omega)} \leq C\|f\|_{L^\infty(M)}. \quad (18)$$

Then, by Moser iteration [19], we can get that

$$\|u\|_{L^\infty(M)} \leq C(\|u\|_{L^2(M)} + \|f\|_{L^\infty(M)}) \leq C\|f\|_{C^\alpha(M)}. \quad (19)$$

This ends the proof. \square

3 Choose the base point

Let's first introduce the space we're going to work on. Define for $0 < \alpha < 1$ and $k \in \mathbb{N}$

$$\mathcal{H}_\omega^{2,\alpha} = \{\varphi \in C^{2,\alpha}(M) | \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \int_M \varphi \omega^n = 0\}, \quad (20)$$

$$C_\omega^{k,\alpha}(M) = \{f \in C^{k,\alpha}(M) | \int_M f \omega^n = 0\}. \quad (21)$$

More generally, θ_φ could be defined on the space $\mathcal{H}_\omega^{2,\alpha}$ if we took the definition as in (9). Therefore, we define, still denoted by θ_φ ,

$$\begin{aligned} \theta : \mathcal{H}_\omega^{2,\alpha} &\rightarrow C^\alpha(M) \\ \varphi &\mapsto \theta_\varphi = -\log \frac{\omega_\varphi^n}{\omega^n} - \underline{R}\varphi + \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega^n + P_\varphi, \end{aligned}$$

where $P_\varphi \in C_\omega^{2,\alpha}(M) \subset C_\omega^\alpha(M)$ is determined by

$$\Delta_\varphi P_\varphi = \text{tr}_\varphi(\text{Ric}(\omega) - \underline{R}\omega), \int_M P_\varphi \omega^n = 0. \quad (22)$$

Define

$$\begin{aligned} F_r &: \mathcal{H}_\omega^{2,\alpha} \rightarrow C_\omega^\alpha(M) \\ \varphi &\mapsto r\theta_\varphi + \varphi. \end{aligned}$$

Denote $\varphi_0 = 0 \in \mathcal{H}_\omega^{2,\alpha}$. As we described the difficulties in the introduction, φ_0 is not enough for our purpose. We need to find "better" base point to apply inverse function theorem.

Let

$$\varphi_1 = \varphi_0 + ru_1 + \frac{r^2}{2}u_2 + \frac{r^3}{6}u_3, \quad (23)$$

where u_i 's are smooth functions on M with $\int_M u_i \omega^n = 0$ that we'll specify later. First we'll expand $F_r(\varphi_1)$ in terms of r at $r = 0$. Denote $u_r = ru_1 + \frac{r^2}{2}u_2 + \frac{r^3}{6}u_3$. Compute

$$\frac{\partial \theta_{\varphi_1}}{\partial r} = -\Delta_{\varphi_1} \dot{u}_r - \underline{R} \dot{u}_r + \int_M \Delta_{\varphi_1} \dot{u}_r \omega^n + \mathcal{D}P|_{\varphi_1}(\dot{u}_r), \quad (24)$$

where $\mathcal{D}P|_{\varphi_1} : C_\omega^{2,\alpha}(M) \rightarrow C_\omega^\alpha(M)$ is the linearization of P_φ at $\varphi = \varphi_1$ and it satisfies

$$\Delta_{\varphi_1}(\mathcal{D}P|_{\varphi_1}(u)) = \langle \partial \bar{\partial} u, \partial \bar{\partial} P_{\varphi_1} - (\text{Ric}(\omega) - \underline{R}\omega) \rangle_{\varphi_1}, \int_M (\mathcal{D}P|_{\varphi_1}(u)) \omega^n = 0. \quad (25)$$

Take one more derivative of θ_{φ_1} , we get

$$\frac{\partial^2 \theta_{\varphi_1}}{\partial^2 r} = -(\Delta_{\varphi_1} \ddot{u}_r - |\partial \bar{\partial} \dot{u}_r|_{\varphi_1}^2) - \underline{R} \ddot{u}_r + \int_M (\Delta_{\varphi_1} \ddot{u}_r - |\partial \bar{\partial} \dot{u}_r|_{\varphi_1}^2) \omega^n + \mathcal{D}P|_{\varphi_1}(\ddot{u}_r) + \left(\frac{\partial}{\partial \varphi} \mathcal{D}P|_{\varphi} \right)|_{\varphi_1}(\dot{u}_r, \dot{u}_r) \quad (26)$$

where the last term is given by the unique solution of the following elliptic equation

$$\begin{aligned} \Delta_{\varphi_1} f &= 2 \langle \partial \bar{\partial} \dot{u}_r, \partial \bar{\partial} (\mathcal{D}P|_{\varphi_1}(\dot{u}_r)) \rangle_{\varphi_1} - \dot{u}_{r,i\bar{p}} \dot{u}_{r,p\bar{j}} (P_{\varphi_1,j\bar{i}} - (\text{Ric}(\omega) - \underline{R}\omega)_{j\bar{i}}) \\ &\quad - \dot{u}_{r,i\bar{p}} \dot{u}_{r,j\bar{i}} (P_{\varphi_1,p\bar{j}} - (\text{Ric}(\omega) - \underline{R}\omega)_{p\bar{j}}) \end{aligned}$$

with $\int_M f \omega^n = 0$. Thus, we get the expansion of $F_r(\varphi_1)$ of r at $r = 0$,

$$F_r(\varphi_1) = r\theta_{\varphi_1} + \varphi_1 \quad (27)$$

$$= \varphi_0 + r(u_1 + \theta_{\varphi_0}) + \frac{r^2}{2}(u_2 + 2\frac{\partial \theta_{\varphi_1}}{\partial r}|_{r=0}) + \frac{r^3}{6}(u_3 + 3\frac{\partial^2 \theta_{\varphi_1}}{\partial^2 r}|_{r=0}) + O(r^4). \quad (28)$$

It suggests that we should define

$$\begin{aligned} u_1 &= -\theta_{\varphi_0} \\ u_2 &= -2\frac{\partial \theta_{\varphi_1}}{\partial r}|_{r=0} = -2\left(-\Delta_{\varphi_0} u_1 - \underline{R}u_1 + \mathcal{D}P|_{\varphi_0}(u_1)\right) \\ u_3 &= -3\frac{\partial^2 \theta_{\varphi_1}}{\partial^2 r}|_{r=0} = -3\left(-\Delta_{\varphi_0} u_2 - \underline{R}u_2 + \mathcal{D}P|_{\varphi_0}(u_2) + |\partial \bar{\partial} u_1|_{\varphi_0}^2 - \int_M |\partial \bar{\partial} u_1|_{\varphi_0}^2 \omega^n \right. \\ &\quad \left. + \left(\frac{\partial}{\partial \varphi} \mathcal{D}P|_{\varphi} \right)|_{\varphi_0}(u_1, u_1)\right) \end{aligned}$$

It's clear from definitions that u_i' s are fixed smooth functions with C^k norm bounds only depend on φ_0 . Therefore, we could choose $r > 0$ sufficiently small such that $\varphi_1 \in \mathcal{H}_\omega^{2,\alpha}$ with $\|\varphi_1\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$. And we expect that $F_r(\varphi_1)$ is r^4 close to "0" in appropriate norms. This observation can be made more precise as the following lemma:

Lemma 3.1. *Notations as described above, for $r > 0$ sufficiently small, we have*

$$\|F_r(\varphi_1)\|_{C^\alpha(M)} \leq Cr^4.$$

Proof. Proof of Lemma 3.1. It suffices to show that

$$\|\theta_{\varphi_1} - (\theta_{\varphi_0} + r \frac{\partial \theta_{\varphi_1}}{\partial r}|_{r=0} + \frac{r^2}{2} \frac{\partial^2 \theta_{\varphi_1}}{\partial^2 r}|_{r=0})\|_{C^\alpha(M)} \leq Cr^3. \quad (29)$$

By Taylor expansion theorem, we could write the remaining error of the function θ_{φ_1} and its second order Taylor expansion as an integral,

$$R(x) = \frac{1}{2!} \int_0^r (r-s)^2 \left(\frac{\partial^3 \theta_{\varphi_1}}{\partial^3 r}|_{r=s}(x) \right) ds. \quad (30)$$

So it suffices to show that for any $s \in [0, r]$ with $r > 0$ sufficiently small

$$\left\| \frac{\partial^3 \theta_{\varphi_1}}{\partial^3 r}|_{r=s} \right\|_{C^\alpha(M)} \leq C. \quad (31)$$

Denote $\varphi_s = \varphi_0 + su_1 + \frac{s^2}{2}u_2 + \frac{s^3}{6}u_3$ and $u_s = su_1 + \frac{s^2}{2}u_2 + \frac{s^3}{6}u_3$. Compute

$$\begin{aligned} \frac{\partial^3 \theta_{\varphi_1}}{\partial^3 r}|_{r=s} &= -(\Delta_{\varphi_s} u_s^{(3)} - 3\langle \partial \bar{\partial} u_s^{(1)}, \partial \bar{\partial} u_s^{(2)} \rangle_{\varphi_s} + 2(\partial \bar{\partial} u_s^{(1)})^{*3}) \\ &\quad - \int_M (\Delta_{\varphi_s} u_s^{(3)} - 3\langle \partial \bar{\partial} u_s^{(1)}, \partial \bar{\partial} u_s^{(2)} \rangle_{\varphi_s} + 2(\partial \bar{\partial} u_s^{(1)})^{*3}) \omega^n - \underline{R} u_s^{(3)} \\ &\quad + \mathcal{D}P|_{\varphi_s}(u_s^{(3)}) + 2\left(\frac{\partial}{\partial \varphi} \mathcal{D}P|_{\varphi}\right)|_{\varphi=\varphi_s}(u_s^{(2)}, u_s^{(1)}) + \left(\frac{\partial}{\partial \varphi} \mathcal{D}P|_{\varphi}\right)|_{\varphi=\varphi_s}(u_s^{(1)}, u_s^{(2)}) \\ &\quad + \left(\frac{\partial^2}{\partial^2 \varphi} \mathcal{D}P|_{\varphi}\right)|_{\varphi=\varphi_s}(u_s^{(1)}, u_s^{(1)}, u_s^{(1)}). \end{aligned}$$

It's obvious that the first two lines has uniform C^α norm as we expected. Therefore, we need to estimate the last four terms of the above equation. Let's first consider P_{φ_s} . It satisfies the Laplacian equation as described in Lemma 2.2, so we get that

$$\|P_{\varphi_s}\|_{C^{2,\alpha}(M)} \leq C. \quad (32)$$

Then we can estimate $\mathcal{D}P|_{\varphi_s}(u)$ using (32) and Lemma 2.2 since it satisfies the similar Laplacian equation with right hand side depending on second order derivatives of P_{φ_s} , we can conclude that

$$\|\mathcal{D}P|_{\varphi_s}(u)\|_{C^{2,\alpha}(M)} \leq C\|u\|_{C^{2,\alpha}(M)}. \quad (33)$$

Thus, we could further estimate the term using the same argument in Lemma 2.2

$$\|(\frac{\partial}{\partial\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(u,v)\|_{C^{2,\alpha}(M)} \leq C\|u\|_{C^{2,\alpha}(M)}\|v\|_{C^{2,\alpha}(M)}. \quad (34)$$

Finally, we could estimate the term $(\frac{\partial^2}{\partial^2\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(u,v,w)$ which satisfies the equation

$$\Delta_{\varphi_s}f = \langle\partial\bar{\partial}w, \partial\bar{\partial}\left((\frac{\partial}{\partial\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(u,v)\right)\rangle_{\varphi_s} + \langle\partial\bar{\partial}u, \partial\bar{\partial}\left((\frac{\partial}{\partial\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(v,w)\right)\rangle_{\varphi_s} \quad (35)$$

$$+ \langle\partial\bar{\partial}v, \partial\bar{\partial}\left((\frac{\partial}{\partial\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(u,w)\right)\rangle_{\varphi_s} + \partial\bar{\partial}v * \partial\bar{\partial}w * \partial\bar{\partial}\left(\mathcal{D}P|_{\varphi_s}(u)\right) \quad (36)$$

$$+ \partial\bar{\partial}u * \partial\bar{\partial}w * \partial\bar{\partial}\left(\mathcal{D}P|_{\varphi_s}(v)\right) + \partial\bar{\partial}v * \partial\bar{\partial}u * \partial\bar{\partial}\left(\mathcal{D}P|_{\varphi_s}(w)\right) \quad (37)$$

$$+ \partial\bar{\partial}u * \partial\bar{\partial}v * \partial\bar{\partial}w * \left(\partial\bar{\partial}P_{\varphi_s} - \text{Ric}(\omega) - \underline{R}\omega\right), \int_M f\omega^n = 0. \quad (38)$$

Thus by the Lemma 2.2, we can conclude that

$$\|(\frac{\partial^2}{\partial^2\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(u,v,w)\| \leq C\|u\|_{C^{2,\alpha}(M)}\|v\|_{C^{2,\alpha}(M)}\|w\|_{C^{2,\alpha}(M)}. \quad (39)$$

Since $\|u_s^{(i)}\|_{C^{2,\alpha}(M)} \leq C$ for $1 \leq i \leq 3$,

$$\|\frac{\partial^3\theta_{\varphi_1}}{\partial^3r}|_{r=s}\|_{C^\alpha(M)} \leq C. \quad (40)$$

Thus it ends the proof of the lemma. \square

4 Proof of Theorem 1.1

In last section, we have shown that for $r > 0$ sufficiently small, $\|F_r(\varphi_1)\|_{C^\alpha(M)} \leq Cr^4$. Next we'll construct a contract map defined on a $r^{1+\epsilon}$ neighborhood of φ_1 in $C^{2,\alpha}$ space, which is similar to the proof of inverse function theorem. Since $F_r(\varphi_1)$ is r^4 small in C^α norm, we could start the iterating process from φ_1 and keep every following term stay within the prescribed $r^{1+\epsilon}$ neighborhood of φ_1 .

First, we have to understand the linearization of $F_r : \mathcal{H}_\omega^{2,\alpha} \rightarrow C_\omega^\alpha(M)$ at $\varphi = \varphi_1$. Compute

$$\begin{aligned} \mathcal{D}F_r|_{\varphi_1} : C_\omega^{2,\alpha}(M) &\rightarrow C_\omega^\alpha(M) \\ u &\mapsto -r\Delta_{\varphi_1}u + (1 - r\underline{R})u + r\left(\int_M (\Delta_{\varphi_1}u)\omega^n + \mathcal{D}P|_{\varphi_1}(u)\right), \end{aligned}$$

where $\mathcal{D}P|_{\varphi_1}(u)$ satisfies

$$\Delta_{\varphi_1}\left(\mathcal{D}P|_{\varphi_1}(u)\right) = \langle\partial\bar{\partial}u, \left(\partial\bar{\partial}P_{\varphi_1} - (\text{Ric}(\omega) - \underline{R}\omega)\right)\rangle_{\varphi_1}, \int_M \left(\mathcal{D}P|_{\varphi_1}(u)\right)\omega^n = 0. \quad (41)$$

We summarize the properties of $\mathcal{D}F_r|_{\varphi_1}$ as the following lemma:

Lemma 4.1. *Suppose $0 < \alpha < 1$. Then, for $r > 0$ sufficiently small, the linearization of $F_r : \mathcal{H}_\omega^{2,\alpha} \rightarrow C_\omega^\alpha(M)$ at $\varphi = \varphi_1$, $\mathcal{D}F_r|_{\varphi_1} : C_\omega^{2,\alpha}(M) \rightarrow C_\omega^\alpha(M)$, is injective and also surjective. Moreover, the operator norm of the inverse of $(\mathcal{D}F_r|_{\varphi_1})$ has the upper bound*

$$\|(\mathcal{D}F_r|_{\varphi_1})^{-1}\| \leq Cr^{-\frac{2-\alpha}{1-\alpha}}.$$

Before proving Lemma 4.1, we'll need the estimate of $\mathcal{D}P|_\varphi(u)$ for $\|\varphi\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$. We summarize it as the following lemma:

Lemma 4.2. *Suppose $\|\varphi\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$, then we have the estimate for any $1 < p < \infty$,*

$$\|(\mathcal{D}P|_\varphi(u))\|_{L^p(M)} \leq C_p \|u\|_{L^p(M)}. \quad (42)$$

Remark. *Since ω and ω_φ are equivalent metrics if $\|\varphi\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$, we make no efforts to distinguish between L^p spaces with respect to the two metrics hereafter.*

Proof. We first introduce the Green function $G_\varphi(x, y)$ of the metric ω_φ . Then we define

$$T(u)(x) = \int_M G_\varphi(x, y) \left(u(P_{\varphi, i\bar{j}} - (\text{Ric}(\omega) - \underline{R}\omega)_{i\bar{j}}) \right)_{, \bar{i}\bar{j}}(y) \omega_\varphi^n \quad (43)$$

$$= \int_M (G_\varphi(x, y))_{, \bar{i}\bar{j}} \left(u(P_{\varphi, i\bar{j}} - (\text{Ric}(\omega) - \underline{R}\omega)_{i\bar{j}}) \right)(y) \omega_\varphi^n. \quad (44)$$

Since

$$\Delta_\varphi(\mathcal{D}P|_\varphi(u)) = \left(u(P_{\varphi, i\bar{j}} - (\text{Ric}(\omega) - \underline{R}\omega)_{i\bar{j}}) \right)_{, \bar{i}\bar{j}}, \int_M (\mathcal{D}P|_\varphi(u)) \omega_\varphi^n = 0, \quad (45)$$

we have

$$\mathcal{D}P|_\varphi(u) = T(u) - \int_M T(u) \omega_\varphi^n \quad (46)$$

For $i, j \in \mathbb{N}$, we define the operator

$$T_{\bar{i}\bar{j}}f = \int_M (G_\varphi(x, y))_{, \bar{i}\bar{j}} f(y) \omega_\varphi^n.$$

$T_{\bar{i}\bar{j}}$ is a Calderon-Zygmund([19]) operator which maps L^p functions to L^p functions for any $1 < p < \infty$. Moreover we can show $T_{\bar{i}\bar{j}}$ has uniform norms. To see this, we consider the Laplacian equation

$$\Delta_\varphi u = f, \int_M u \omega_\varphi^n = 0. \quad (47)$$

Thus, we see that the solution satisfies

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j} u(x) = (T_{\bar{i}\bar{j}}f)(x). \quad (48)$$

So it suffices to show the uniform $W^{2,2}$ estimates of (47), which follows from the fact that $\|\varphi\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$ and the standard L^p theory of elliptic equation([19]). We have the estimate for any $p \in (1, +\infty)$

$$\|T_{ij}f\|_{L^p(M)} \leq C_p \|f\|_{L^p(M)}. \quad (49)$$

Thus, taking advantages of the above estimate, we can get

$$\|T(u)\|_{L^p(M)} \leq \sum_{k,l} C_p \|u(g_{\varphi}^{i\bar{l}} g_{\varphi}^{k\bar{j}} (P_{\varphi, i\bar{j}} - (\text{Ric}(\omega) - \underline{R}\omega)_{i\bar{j}}))\|_{L^p(M)} \quad (50)$$

$$\leq C_p \|u\|_{L^p(M)}. \quad (51)$$

Thus, we have for any $1 < p < \infty$

$$\|(\mathcal{D}P|_{\varphi}(u))\|_{L^p(M)} \leq C_p \|u\|_{L^p(M)}. \quad (52)$$

This ends the proof of Lemma 4.2. \square

Now we can prove Lemma 4.1.

Proof. Proof of Lemma 4.1. First we show that $\mathcal{D}F_r|_{\varphi_1}$ is injective. Suppose there exists $u \in C_{\omega}^{2,\alpha}(M)$ such that

$$-r\Delta_{\varphi_1}u + (1 - r\underline{R})u + r\left(\int_M (\Delta_{\varphi_1}u)\omega^n + \mathcal{D}P|_{\varphi_1}(u)\right) = 0. \quad (53)$$

It suffices to show that $u = 0$. Multiply u on both hand sides of (53) and integrate against $\omega_{\varphi_1}^n$.

$$0 = r \int_M |\nabla u|_{\varphi_1}^2 \omega_{\varphi_1}^n + (1 - r\underline{R}) \int_M u^2 \omega_{\varphi_1}^n + r \left(\int_M (\Delta_{\varphi_1}u)\omega^n \right) \left(\int_M u \omega_{\varphi_1}^n \right) + r \int_M (\mathcal{D}P|_{\varphi_1}(u)) u \omega_{\varphi_1}^n \quad (54)$$

$$\geq (1 - r\underline{R}) \int_M u^2 \omega_{\varphi_1}^n + r \left\{ \left(\int_M (\Delta_{\varphi_1}u)\omega^n \right) \left(\int_M u \omega_{\varphi_1}^n \right) + \int_M (\mathcal{D}P|_{\varphi_1}(u)) u \omega_{\varphi_1}^n \right\}. \quad (55)$$

We focus on estimates of the later two terms in (55). Consider

$$\begin{aligned} \int_M (\Delta_{\varphi_1}u)\omega^n &= \int_M (\Delta_{\varphi_1}u) \left(\frac{\omega^n}{\omega_{\varphi_1}^n} \right) \omega_{\varphi_1}^n = \int_M u \left(\Delta_{\varphi_1} \frac{\omega^n}{\omega_{\varphi_1}^n} \right) \omega_{\varphi_1}^n \\ &= \int_M u \frac{\omega^n}{\omega_{\varphi_1}^n} g_{\varphi_1}^{i\bar{j}} \left(-g_{\varphi_1}^{k\bar{l}} \varphi_{1, k\bar{l}i\bar{j}} + g_{\varphi_1}^{k\bar{p}} g_{\varphi_1}^{q\bar{l}} \varphi_{1, \bar{p}q\bar{j}} \varphi_{1, k\bar{l}i} + g_{\varphi_1}^{\bar{p}q} g_{\varphi_1}^{k\bar{l}} \varphi_{1, \bar{p}q\bar{j}} \varphi_{1, k\bar{l}i} \right) \omega_{\varphi_1}^n \\ &\geq -Cr \left(\int_M |u| \omega_{\varphi_1}^n \right) \end{aligned}$$

where the derivatives are covariant derivatives of ω . Thus, we have that

$$r \left(\int_M (\Delta_{\varphi_1}u)\omega^n \right) \left(\int_M u \omega_{\varphi_1}^n \right) \geq -Cr^2 \int_M u^2 \omega_{\varphi_1}^n \quad (56)$$

To estimate the last term of (55), we need the following estimate of $(\mathcal{D}P|_{\varphi_1})$.

Choosing $r > 0$ sufficiently small, we can get $\|\varphi_1\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$. Using Lemma 4.2, we get

$$\|(\mathcal{D}P|_{\varphi_1}(u))\|_{L^2(M)} \leq C\|u\|_{L^2(M)}. \quad (57)$$

Thus for the last term in (55) we have the estimate

$$\int_M (\mathcal{D}P|_{\varphi_1}(u))u\omega_{\varphi_1}^n \geq -C \int_M u^2\omega_{\varphi_1}^n. \quad (58)$$

Therefore, combining the estimates above, we have that

$$0 \geq (1 - Cr) \int_M u^2\omega_{\varphi_1}^n. \quad (59)$$

It implies that when $r > 0$ sufficiently small, we have that $u = 0$. So we have proved the injectivity of $(\mathcal{D}F_r|_{\varphi_1})$.

Next, we show the surjectivity of $(\mathcal{D}F_r|_{\varphi_1})$ and the upper bound of $\|(\mathcal{D}F_r|_{\varphi_1})^{-1}\|$ together. For $f \in C_\omega^\alpha(M)$, we'll use continuity method to solve the equation

$$\mathcal{D}F_r|_{\varphi_1}(u) = f. \quad (60)$$

Define for $s \in [0, 1]$,

$$L_s : C^{2,\alpha}(M) \rightarrow C^\alpha(M) \quad (61)$$

$$u \mapsto -r\Delta_{\varphi_1}u + (1 - r\underline{R})u + sr\left(\int_M (\Delta_{\varphi_1}u)\omega^n + \mathcal{D}P|_{\varphi_1}(u)\right). \quad (62)$$

First, we show that for any $s \in [0, 1]$

$$\|u\|_{C^{2,\alpha}(M)} \leq C_r\|L_s u\|_{C^\alpha(M)}. \quad (63)$$

From the definition of L_s , we get that,

$$\Delta_{\varphi_1}u = -\frac{1}{r}L_s u + \frac{1 - r\underline{R}}{r}u + s\left(\int_M (\Delta_{\varphi_1}u)\omega^n + \mathcal{D}P|_{\varphi_1}(u)\right). \quad (64)$$

Since we choose $r > 0$ sufficiently small s.t. $\|\varphi_1\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$, we can get from Schauder estimate,

$$\begin{aligned} \|u\|_{C^{2,\alpha}(M)} &\leq C\left(\|\Delta_{\varphi_1}u\|_{C^\alpha(M)} + \|u\|_{L^\infty(M)}\right) \\ &\leq C\left(\frac{1}{r}\|L_s u\|_{C^\alpha(M)} + \frac{1}{r}\|u\|_{C^\alpha(M)} + \left|\int_M (\Delta_{\varphi_1}u)\omega^n\right| + \|(\mathcal{D}P|_{\varphi_1}(u))\|_{C^\alpha(M)} + \|u\|_{L^\infty(M)}\right) \\ &\leq C_0\left(\frac{1}{r}\|L_s u\|_{C^\alpha(M)} + \frac{1}{r}\|u\|_{C^\alpha(M)} + \|(\mathcal{D}P|_{\varphi_1}(u))\|_{C^\alpha(M)} + \|u\|_{L^\infty(M)}\right). \end{aligned}$$

By interpolations [19], we have

$$\|u\|_{C^\alpha(M)} \leq \frac{r}{4C_0}\|u\|_{C^{2,\alpha}(M)} + Cr^{-\frac{\alpha}{1-\alpha}}\|u\|_{L^\infty(M)}. \quad (65)$$

Also, for term $\|\mathcal{D}P|_{\varphi_1}(u)\|_{C^\alpha(M)}$, since it satisfies equation (41), we have estimate

$$\|\mathcal{D}P|_{\varphi_1}(u)\|_{C^\alpha(M)} \leq C\|\langle \partial\bar{\partial}u, (\partial\bar{\partial}P_{\varphi_1} - (\text{Ric}(\omega) - \underline{R}\omega)) \rangle_{\varphi_1}\|_{L^\infty(M)} \quad (66)$$

$$\leq C\|\partial\bar{\partial}u\|_{L^\infty(M)} \leq \frac{1}{4C_0}\|u\|_{C^{2,\alpha}(M)} + C\|u\|_{L^\infty(M)}. \quad (67)$$

Combining estimates of (65) and (66), we have that

$$\|u\|_{C^{2,\alpha}(M)} \leq C\left(\frac{1}{r}\|L_s u\|_{C^\alpha(M)} + r^{-\frac{1}{1-\alpha}}\|u\|_{L^\infty(M)}\right). \quad (68)$$

Now we focus on estimates of $\|u\|_{L^\infty(M)}$. For $p > 1$, We could first multiply $|u|^p$ on both hand sides of (64) and integrate against $\omega_{\varphi_1}^n$ on the region $\{u > 0\}$. Then we'll get by a similar argument which we use to prove the injectivity,

$$\begin{aligned} \frac{1}{r} \int_{u>0} (L_s u) u^p \omega_{\varphi_1}^n &\geq \int_{u>0} p u^{p-1} |\nabla u|_{\varphi_1}^2 \omega_{\varphi_1} + \frac{1-r\underline{R}}{r} \int_{u>0} u^{p+1} \omega_{\varphi_1}^n - Cr \left(\int_M |u| \omega_{\varphi_1}^n \right) \left(\int_{u>0} u^p \omega_{\varphi_1}^n \right) \\ &\quad - C_p \|u\|_{L^{p+1}(M)} \left(\int_{u>0} u^{p+1} \omega_{\varphi_1} \right)^{\frac{p}{p+1}} \\ &\geq \frac{1-r\underline{R}}{r} \int_{u>0} u^{p+1} \omega_{\varphi_1}^n - C_p \int_M u^{p+1} \omega_{\varphi_1}^n. \end{aligned}$$

Multiply $|u|^p$ on both hand sides of (64) and integrate against $\omega_{\varphi_1}^n$ on the region $\{u < 0\}$. Similarly we get

$$-\frac{1}{r} \int_{u<0} (L_s u) |u|^p \omega_{\varphi_1}^n \geq \frac{1-r\underline{R}}{r} \int_{u<0} |u|^{p+1} \omega_{\varphi_1}^n - C_p \int_M |u|^{p+1} \omega_{\varphi_1}^n. \quad (69)$$

Thus, we get for $p < p_0 < \infty$, we could choose our $r > 0$ small such that

$$\frac{1}{r} \int_M |u|^{p+1} \omega_{\varphi_1}^n \leq \frac{1}{r} \|u\|_{L^{p+1}(M)}^{\frac{p}{p+1}} \|L_s u\|_{L^{p+1}(M)}. \quad (70)$$

And then for $p < p_0 + 1$

$$\|u\|_{L^p(M)} \leq C\|L_s u\|_{L^p(M)} \leq C\|L_s u\|_{L^\infty(M)}. \quad (71)$$

By L^p theory of elliptic equation for (64), we get for $p < p_0 + 1$

$$\|u\|_{W^{2,p}(M)} \leq C\left(\frac{1}{r}\|L_s u\|_{L^p(M)} + \frac{1}{r}\|u\|_{L^p(M)}\right) \quad (72)$$

$$\leq \frac{C}{r}\|L_s u\|_{L^\infty(M)}. \quad (73)$$

By sobolev embedding, we can get that for $p > n$

$$\|u\|_{L^\infty(M)} \leq C\|u\|_{W^{2,p}(M)} \leq \frac{C}{r}\|L_s u\|_{L^\infty(M)}. \quad (74)$$

Therefore, we conclude that

$$\|u\|_{C^{2,\alpha}(M)} \leq Cr^{-\frac{2-\alpha}{1-\alpha}}\|L_s u\|_{C^\alpha(M)} \quad (75)$$

Since the norm is independent of $s \in [0, 1]$ and obviously $L_0 : C^{2,\alpha}(M) \rightarrow C^\alpha(M)$ is onto, thus by continuity method in [19], we conclude that $L_1 : C^{2,\alpha}(M) \rightarrow C^\alpha(M)$ is also onto. Thus we have shown that $\mathcal{D}F_r|_{\varphi_1} = L_1$ is surjective. And

$$\|(\mathcal{D}F_r|_{\varphi_1})^{-1}(f)\|_{C^{2,\alpha}(M)} \leq Cr^{-\frac{2-\alpha}{1-\alpha}}\|f\|_{C^\alpha(M)}. \quad (76)$$

This ends the proof of Lemma 4.1. \square

Define functional Ψ in a $C^{2,\alpha}$ -neighborhood of φ_1 as

$$\begin{aligned} \Psi : \mathcal{H}_\omega^{2,\alpha} &\rightarrow C_\omega^{2,\alpha}(M) \\ \varphi &\mapsto \varphi + (\mathcal{D}F_r|_{\varphi_1})^{-1}(-F_r(\varphi)) \end{aligned}$$

Our goal is to find $\varphi \in \mathcal{H}_\omega^{2,\alpha}$ such that $F_r(\varphi) = 0$. Given the definition of Ψ , our problem comes down to find the fixed point of Ψ . So we need to show that Ψ is a contraction in a small neighborhood of $\varphi_1 \in \mathcal{H}_\omega^{2,\alpha}$.

Lemma 4.3. *There exists some $\delta > 0$, such that if $\varphi, \tilde{\varphi} \in \mathcal{H}_\omega^{2,\alpha}$ with $\|\varphi - \varphi_1\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}}\delta$ and $\|\tilde{\varphi} - \varphi_1\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}}\delta$, then*

$$\|\Psi(\varphi) - \Psi(\tilde{\varphi})\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}\|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)}. \quad (77)$$

Proof. Denote $\varphi_s = s\varphi + (1-s)\tilde{\varphi}$. Suppose $\|\varphi - \varphi_1\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}}\delta$ and $\|\tilde{\varphi} - \varphi_1\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}}\delta$. We'll specify $\delta > 0$ later.

We have

$$\begin{aligned} \Psi(\varphi) - \Psi(\tilde{\varphi}) &= \int_0^1 \frac{\partial}{\partial s} \Psi(\varphi_s) ds \\ &= (\varphi - \tilde{\varphi}) - \int_0^1 (\mathcal{D}F_r|_{\varphi_1})^{-1}(\mathcal{D}F_r|_{\varphi_s}(\varphi - \tilde{\varphi})) ds \\ &= - \int_0^1 (\mathcal{D}F_r|_{\varphi_1})^{-1} \{(\mathcal{D}F_r|_{\varphi_s} - \mathcal{D}F_r|_{\varphi_1})(\varphi - \tilde{\varphi})\} ds. \end{aligned}$$

We consider the term

$$\begin{aligned} (\mathcal{D}F_r|_{\varphi_s} - \mathcal{D}F_r|_{\varphi_1})(\varphi - \tilde{\varphi}) &= r \left(-(\Delta_{\varphi_s} - \Delta_{\varphi_1})(\varphi - \tilde{\varphi}) + \int_M ((\Delta_{\varphi_s} - \Delta_{\varphi_1})(\varphi - \tilde{\varphi})) \omega^n \right. \\ &\quad \left. + (\mathcal{D}P|_{\varphi_s} - \mathcal{D}P|_{\varphi_1})(\varphi - \tilde{\varphi}) \right). \end{aligned}$$

Thus, we know that

$$\|(\mathcal{D}F_r|_{\varphi_s} - \mathcal{D}F_r|_{\varphi_1})(\varphi - \tilde{\varphi})\|_{C^\alpha(M)} \quad (78)$$

$$\leq Cr \left(r^{\frac{1}{1-\alpha}}\delta \|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)} + \|(\mathcal{D}P|_{\varphi_s} - \mathcal{D}P|_{\varphi_1})(\varphi - \tilde{\varphi})\|_{C^\alpha(M)} \right). \quad (79)$$

By definitions of $\mathcal{D}P|_\varphi$ in (41), we have

$$\Delta_{\varphi_1}(\mathcal{D}P|_{\varphi_1}(u)) = \langle \partial \bar{\partial} u, (\partial \bar{\partial} P_{\varphi_1} - (\text{Ric}(\omega) - \underline{R}\omega)) \rangle_{\varphi_1}, \int_M (\mathcal{D}P|_{\varphi_1}(u)) \omega^n = 0 \quad (80)$$

$$\Delta_{\varphi_s}(\mathcal{D}P|_{\varphi_s}(u)) = \langle \partial \bar{\partial} u, (\partial \bar{\partial} P_{\varphi_s} - (\text{Ric}(\omega) - \underline{R}\omega)) \rangle_{\varphi_s}, \int_M (\mathcal{D}P|_{\varphi_s}(u)) \omega^n = 0. \quad (81)$$

So

$$\begin{aligned}
\Delta_{\varphi_1}(\mathcal{D}P|_{\varphi_1}(u) - \mathcal{D}P|_{\varphi_s}(u)) &= \langle \partial \bar{\partial} u, (\partial \bar{\partial} P_{\varphi_1} - (\text{Ric}(\omega) - \underline{R}\omega)) \rangle_{\varphi_1} - \langle \partial \bar{\partial} u, (\partial \bar{\partial} P_{\varphi_s} - (\text{Ric}(\omega) - \underline{R}\omega)) \rangle_{\varphi_s} \\
&\quad + (\Delta_{\varphi_s} - \Delta_{\varphi_1})(\mathcal{D}P|_{\varphi_s}(u)) \\
&= u_{,i\bar{j}}(g_{\varphi_1}^{i\bar{l}} g_{\varphi_1}^{k\bar{j}} - g_{\varphi_s}^{i\bar{l}} g_{\varphi_s}^{k\bar{j}}) P_{\varphi_1, k\bar{l}} + u_{,i\bar{j}} g_{\varphi_s}^{i\bar{l}} g_{\varphi_s}^{k\bar{j}} (P_{\varphi_1} - P_{\varphi_s}) \\
&\quad - u_{,i\bar{j}}(g_{\varphi_1}^{i\bar{l}} g_{\varphi_1}^{k\bar{j}} - g_{\varphi_s}^{i\bar{l}} g_{\varphi_s}^{k\bar{j}}) (\text{Ric}(\omega) - \underline{R}\omega)_{k\bar{l}} + (g_{\varphi_s}^{k\bar{l}} - g_{\varphi_1}^{k\bar{l}}) (\mathcal{D}P|_{\varphi_s}(u))_{,k\bar{l}}
\end{aligned}$$

Thus, by schauder estimate and previous estimate about P_φ and $\mathcal{D}P|_\varphi(u)$ in Section 3,

$$\begin{aligned}
\|\mathcal{D}P|_{\varphi_1}(u) - \mathcal{D}P|_{\varphi_s}(u)\|_{C^{2,\alpha}(M)} &\leq Cr^{\frac{1}{1-\alpha}} \delta (\|u\|_{C^{2,\alpha}(M)} + \|\mathcal{D}P|_{\varphi_s}(u)\|_{C^{2,\alpha}(M)}) \\
&\quad + C\|P_{\varphi_1} - P_{\varphi_s}\|_{C^{2,\alpha}(M)} \|u\|_{C^{2,\alpha}(M)} \\
&\leq Cr^{\frac{1}{1-\alpha}} \delta \|u\|_{C^{2,\alpha}(M)} + C\|P_{\varphi_1} - P_{\varphi_s}\|_{C^{2,\alpha}(M)} \|u\|_{C^{2,\alpha}(M)}.
\end{aligned}$$

Since we have

$$\Delta_{\varphi_1}(P_{\varphi_1} - P_{\varphi_s}) = (g_{\varphi_s}^{k\bar{l}} - g_{\varphi_1}^{k\bar{l}}) P_{\varphi_s, k\bar{l}} + (g_{\varphi_1}^{k\bar{l}} - g_{\varphi_s}^{k\bar{l}}) (\text{Ric}(\omega) - \underline{R}\omega)_{k\bar{l}},$$

then

$$\|P_{\varphi_1} - P_{\varphi_s}\|_{C^{2,\alpha}(M)} \leq Cr^{\frac{1}{1-\alpha}} \delta. \quad (82)$$

Thus, we have

$$\|(\mathcal{D}P|_{\varphi_1} - \mathcal{D}P|_{\varphi_s})(\varphi - \tilde{\varphi})\|_{C^\alpha(M)} \leq \|(\mathcal{D}P|_{\varphi_1} - \mathcal{D}P|_{\varphi_s})(\varphi - \tilde{\varphi})\|_{C^{2,\alpha}(M)} \leq Cr^{\frac{1}{1-\alpha}} \delta \|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)}. \quad (83)$$

By Lemma 4.1, we have that

$$\|(\mathcal{D}F_r|_{\varphi_1})^{-1} \{(\mathcal{D}F_r|_{\varphi_s} - \mathcal{D}F_r|_{\varphi_1})(\varphi - \tilde{\varphi})\}\|_{C^{2,\alpha}(M)} \leq Cr^{-\frac{2-\alpha}{1-\alpha}} r r^{\frac{1}{1-\alpha}} \delta \|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)} \quad (84)$$

$$\leq C\delta \|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)}. \quad (85)$$

And then

$$\|\Psi(\varphi) - \Psi(\tilde{\varphi})\|_{C^{2,\alpha}(M)} \leq C\delta \|\varphi - \tilde{\varphi}\|_{C^{2,\alpha}(M)}. \quad (86)$$

We could choose $\delta > 0$ sufficiently small such that $C\delta < \frac{1}{2}$, and thus it ends the proof of Lemma 4.3. \square

Now we're ready to prove the Theorem 2.1.

Proof. Denote the constant $\delta > 0$ in Lemma 4.3 as δ_0 . Define for $k \in \mathbb{Z}$

$$\varphi_k = \Psi^{k-1}(\varphi_1). \quad (87)$$

Ultimately, we want to show that $\varphi_k \rightarrow \varphi_\infty$ in $C^{2,\alpha}(M)$ norm for some $\varphi_\infty \in \mathcal{H}_\omega^{2,\alpha}$ as $k \rightarrow \infty$. We choose the start point to be φ_1 , thus we need to show that φ_2 stays in the neighborhood of φ_1 for Ψ to be contraction. Compute

$$\begin{aligned}\|\varphi_2 - \varphi_1\|_{C^{2,\alpha}(M)} &= \|(\mathcal{D}F_r|_{\varphi_1})^{-1}(-F_r(\varphi_1))\|_{C^{2,\alpha}(M)} \\ &\leq Cr^{-\frac{2-\alpha}{1-\alpha}}\|F_r(\varphi_1)\|_{C^\alpha(M)} \\ &\leq (Cr^{\frac{1-3\alpha}{1-\alpha}})r^{\frac{1}{1-\alpha}}.\end{aligned}$$

where we use Lemma 3.1 and Lemma 4.1. It's obvious we could choose $\alpha = \frac{1}{4}$ and $r > 0$ sufficiently small such that

$$\|\varphi_2 - \varphi_1\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}r^{\frac{1}{1-\alpha}}\delta_0. \quad (88)$$

By induction, we could get that for any $k \in \mathbb{Z}$

$$\|\varphi_k - \varphi_1\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}}\delta_0, \quad (89)$$

and

$$\|\varphi_{k+1} - \varphi_k\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}\|\varphi_k - \varphi_{k-1}\|_{C^{2,\alpha}(M)} \leq \left(\frac{1}{2}\right)^k r^{\frac{1}{1-\alpha}}\delta_0. \quad (90)$$

Thus, we conclude that there exists some $\varphi_\infty \in \mathcal{H}_\omega^{2,\alpha}$ such that $\varphi_k \rightarrow \varphi_\infty$ in $C^{2,\alpha}(M)$ as $k \rightarrow \infty$. Thus, we get

$$F_r(\varphi_\infty) = 0. \quad (91)$$

From the regularity of elliptic equation, we could immediately see that $\varphi_\infty \in C^\infty(M)$. Also it's clear that

$$\|\varphi_\infty\|_{C^{2,\alpha}(M)} \leq \|\varphi_1\|_{C^{2,\alpha}(M)} + r^{\frac{1}{1-\alpha}}\delta_0 \leq Cr \rightarrow 0, \text{ as } r \rightarrow 0. \quad (92)$$

Then we finish the proof of Theorem 2.1. □

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